Formal Stability Analysis of Optical Resonators

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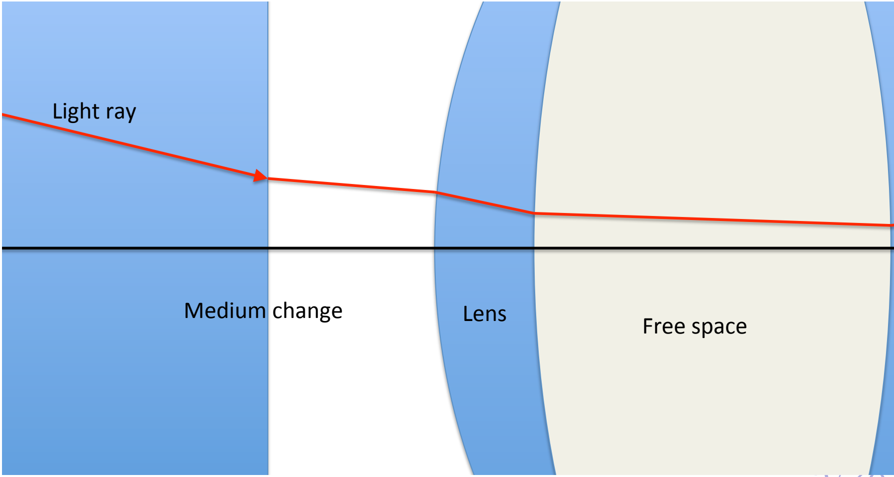
In a few words

- Interactive theorem proving to model optical systems
- More precisely: prove the stability of optical resonators

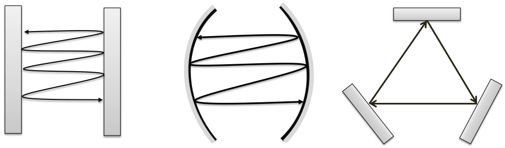
Concrete example

Optical system:

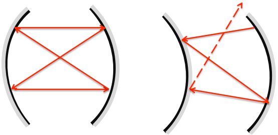


Concrete example

Optical resonators:



Plane- Mirror   Spherical -Mirror   Ring-Mirror



Stable   Unstable

Why optics?

- Optics is used more and more in various technologies
- Very present in critical applications
  (aerospace, military, health, etc.)
- Complex to verify

+ it's new and cool to formalize physics

Geometrical optics

Main characteristics:

- Light is a ray
- "Paraxial" approximation = objects bigger than wavelength $\Rightarrow$ no diffraction
- Fermat principle = use the shortest path

Additional assumption:

Small angles w.r.t. axis (concretely: $\sin(\theta) \approx \theta$)

Related work

- Existing numerical software
  (e.g., reZonator, LASCAD, CODE V)
- Existing computer algebra software
  (e.g., Optica)
- But no existing use of formal methods

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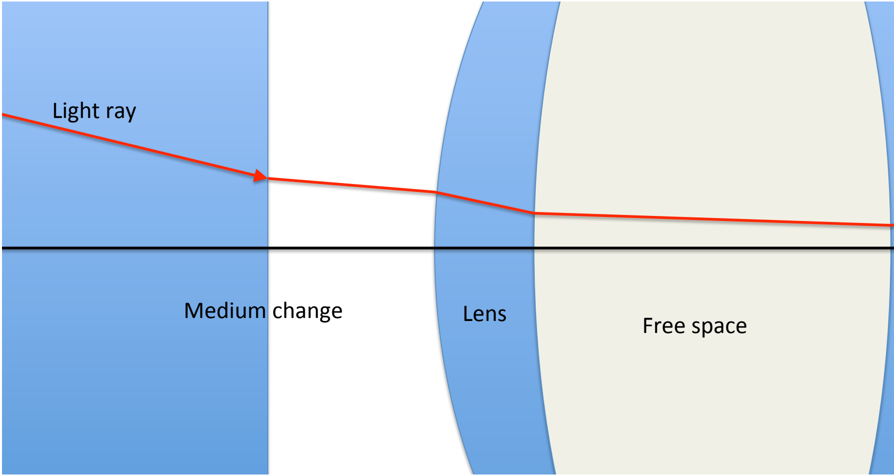
Formalization outline

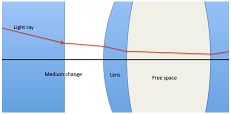
All the following has to be formalized:

- Optical systems
- Light and its interaction with systems
- Matrix representation of light behavior
- Stability analysis

Formalization of optical systems (1/2)



Formalization of optical systems (2/2)



Type definitions:

- Free space: refractive index + distance
  $\rightarrow$ type free_space $\stackrel{\text{def}}{=} \mathbb{R} \times \mathbb{R}$
- Interface between mediums: planar OR spherical
  $\rightarrow$ type interface $\stackrel{\text{def}}{=}$ plane | spherical($\mathbb{R}$)
- Observed ray behaviour: reflected OR transmitted
  $\rightarrow$ type behaviour $\stackrel{\text{def}}{=}$ reflected | transmitted
- Optical system: list of free space + interface + behaviour
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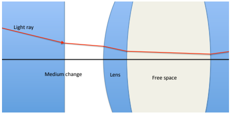

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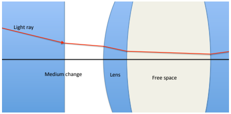

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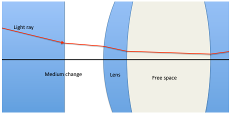

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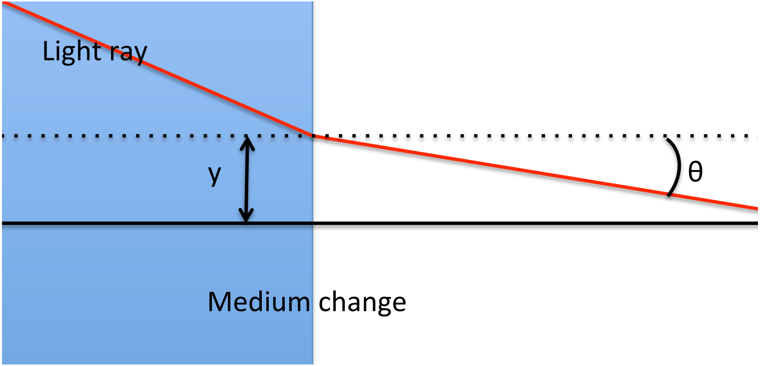
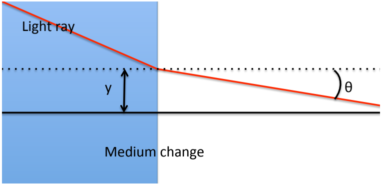
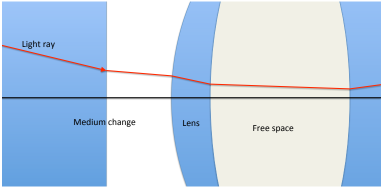
Formalization of light as a ray (1/3)



Formalization of light as a ray (2/3)

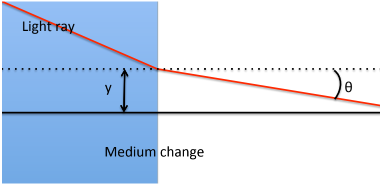


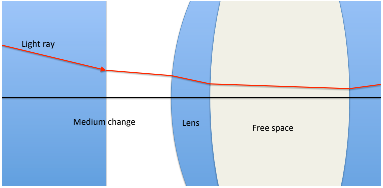


Type definitions:

- Ray at a given point: distance from axis + angle
  $\rightarrow$ `type ray_at_point` $\stackrel{\text{def}}{=} \mathbb{R} \times \mathbb{R}$
- Ray course through a whole system:
  a ray description for every point of the system
  $\rightarrow$ `type ray` $\stackrel{\text{def}}{=}$ `(ray_at_point` $\times$ `ray_at_point) list`

Formalization of light as a ray (2/3)





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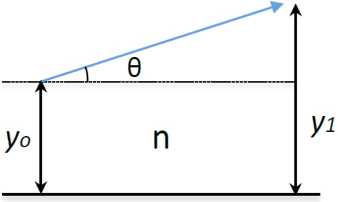
Formalization of light as a ray (3/3)

- Ray in a free space

$$\begin{array}{l} \texttt{is\_valid\_ray\_at\_free\_space} \\ \qquad (y_0, \theta_0) : \texttt{ray\_at\_point} \\ \qquad (y_1, \theta_1) : \texttt{ray\_at\_point} \\ \qquad (n, d) : \texttt{free\_space} \\ \qquad \overset{\text{def}}{\Leftrightarrow} y_1 = y_0 + d * \theta_0 \wedge \theta_1 = \theta_0 \end{array}$$



- Ray transmitted at a plane interface (Snell's law)

$\texttt{is\_valid\_ray\_at\_interface}$ (y_0, θ_0) (y_1, θ_1) n_0 n_1 $\texttt{plane transmitted}$

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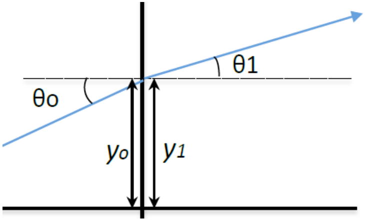

- Similar for reflected (law of reflection)

Formalization of light as a ray (3/3)

- Ray in a free space

$$\texttt{is\_valid\_ray\_at\_free\_space}\ (y_0, \theta_0)\ (y_1, \theta_1)\ (n, d) \overset{\text{def}}{\Leftrightarrow} y_1 = y_0 + d * \theta_0 \wedge \theta_1 = \theta_0$$

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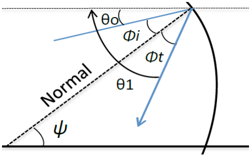
- Similar for reflected (law of reflection)
- Similar for spherical interfaces



→ complex expression, but easy (and boring) underlying reasoning

Formalization of the transfer matrices

- Small angle approximation
  $\rightarrow$ the mapping $y_0 \mapsto y_1$, $\theta_0 \mapsto \theta_1$ is linear

Formalization of the transfer matrices

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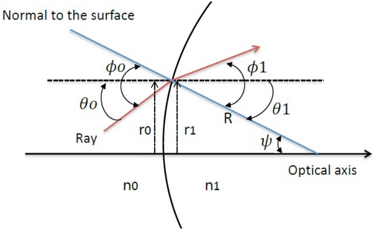
  $\longrightarrow$ the mapping $y_0 \mapsto y_1$, $\theta_0 \mapsto \theta_1$ is linear
- 2×2 matrix for each component

  $\longrightarrow$ example:



$$\begin{bmatrix} y_1 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n_0 - n_1}{n_1 R} & \frac{n_0}{n_1} \end{bmatrix} \begin{bmatrix} y_0 \\ \theta_0 \end{bmatrix}$$

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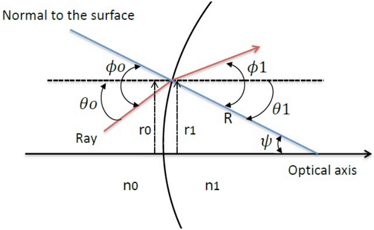

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- Matrix multiplication $\Rightarrow$ matrix for whole system

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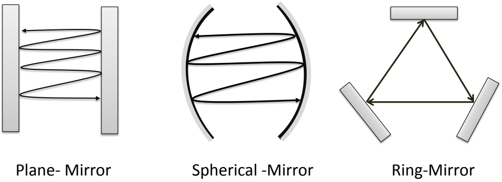
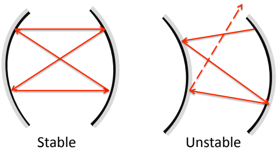
Stability of optical resonators





Formal expression of stability

- A resonator is *stable* if the ray trajectory is bounded after any number of round-trips
- Amounts to say that the vector $(ray, angle)$ is bounded even after raising the matrix of the system to any power:

Definition

$\vdash \forall$ `M. stable_optical_system M` $\Leftrightarrow$

$$\forall y, \theta.\ \exists B.\ \forall n.\ abs\left(M^n * \begin{bmatrix} y \\ \theta \end{bmatrix}\right) \leq B$$

$(abs, \leq\ =$ component-wise$)$

Sylvester's Theorem

Key theorem providing a sufficient condition for stability:

Theorem (Sylvester's Theorem)

$$\vdash \forall\ \mathtt{N\ A\ B\ C\ D}.\ \begin{vmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{vmatrix} = 1 \wedge -1 < \frac{\mathtt{A+D}}{2} \wedge \frac{\mathtt{A+D}}{2} < 1 \implies$$

$$\mathtt{let}\ \theta = \mathtt{acs}\left(\frac{\mathtt{A+D}}{2}\right)\ \mathtt{in}$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{\mathtt{N}} = \frac{1}{\mathtt{sin}(\theta)} \begin{bmatrix} \mathtt{A}*\mathtt{sin}(\mathtt{N}\theta) - \mathtt{sin}((\mathtt{N}-1)\theta) & \mathtt{B}*\mathtt{sin}(\mathtt{N}\theta) \\ \mathtt{C}*\mathtt{sin}(\mathtt{N}\theta) & \mathtt{D}*\mathtt{sin}(\mathtt{N}\theta) - \mathtt{sin}((\mathtt{N}-1)\theta) \end{bmatrix}$$

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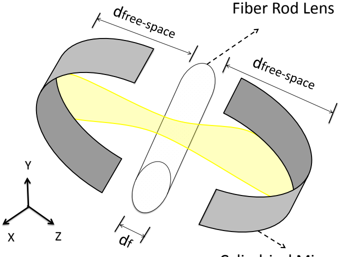
Generalized stability theorem

Sylvester's theorem allows to derive a simple criterion to decide the stability of resonator:

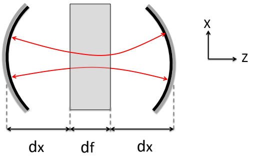
Theorem (Generalized Stability Theorem)

$$\vdash \forall\ \mathtt{A\ B\ C\ D}.\ \begin{vmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{vmatrix} = 1 \wedge -1 < \frac{\mathtt{A+D}}{2} \wedge \frac{\mathtt{A+D}}{2} < 1 \Longrightarrow \mathtt{stable\_optical\_system} \begin{bmatrix} \mathtt{A} & \mathtt{B} \\ \mathtt{C} & \mathtt{D} \end{bmatrix}$$

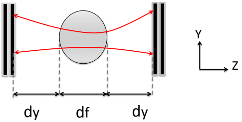
Application: Fabry Perot resonator



(a)



(b)



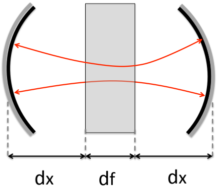
(c)

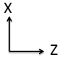
Fabry Perot resonator: formalization in XZ plane

Formal Model of FP Resonator in XZ Plane:

Definition

```
⊢ ∀ R dx nf df.  FP_XZ R dx df nf =
   ([(1,0),spherical R,reflected;
   (1,dx),plane,transmitted;
   (nf,df),plane,transmitted],1,dx)
```





Fabry Perot resonator: stability

Ray-transfer matrix for one round-trip in the resonator:

Theorem

$$\vdash \forall\ \mathtt{R\ dx\ df\ nf.}\quad \mathtt{R} \neq \mathtt{0} \wedge \mathtt{0} < \mathtt{dx} \wedge \mathtt{0} < \mathtt{df} \wedge \mathtt{0} < \mathtt{nf}$$

$$\Longrightarrow \quad \mathtt{system\_composition\ (FP\_XZ\ R\ dx\ df\ nf)} = \begin{bmatrix} 1 - \frac{\mathtt{2} * (\mathtt{df} + \mathtt{2} * \mathtt{dx} * \mathtt{nf})}{\mathtt{nf} * \mathtt{R}} & \mathtt{2} * \mathtt{dx} + \frac{\mathtt{df}}{\mathtt{nf}} \\ -\frac{\mathtt{2}}{\mathtt{R}} & 1 \end{bmatrix}$$

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$$\Longrightarrow \quad \mathtt{system\_composition\ (FP\_XZ\ R\ dx\ df\ nf)} =$$
$$\begin{bmatrix} 1 - \frac{2 * (\mathtt{df} + 2 * \mathtt{dx} * \mathtt{nf})}{\mathtt{nf} * \mathtt{R}} & 2 * \mathtt{dx} + \frac{\mathtt{df}}{\mathtt{nf}} \\ -\frac{2}{\mathtt{R}} & 1 \end{bmatrix}$$

Sufficient conditions for stability:

Theorem (Stability in XZ plane)

$$\vdash \forall\ \mathtt{R\ dx\ df\ nf.}\quad \mathtt{R} \neq 0 \wedge 0 < \mathtt{dx} \wedge 0 < \mathtt{df} \wedge 0 < \mathtt{nf}$$
$$0 < \frac{2*\mathtt{dx}+\frac{\mathtt{df}}{\mathtt{nf}}}{\mathtt{R}} \wedge \frac{2*\mathtt{dx}+\frac{\mathtt{df}}{\mathtt{nf}}}{\mathtt{R}} < 2 \Longrightarrow \mathtt{stable\_optical\_system}$$
$$\mathtt{(system\_composition\ (FP\_XZ\ R\ dx\ df\ nf))}$$

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Conclusion

Formalization of:

- Optical systems
- Behavior of light in geometric optics
- Ray-transfer matrix in paraxial approximation
- Stability analysis
- Application to a recently developped resonator

Future work

- Formal treatment of small angle approximation
- Derivation of Snell's law and law of reflection from Fermat's principle
- Handle more complex models (e.g., misaligned components)

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Thanks!

Questions?

PS: Just in case, looking for a job in Germany, preferably close to Stuttgart... ;-)